

Entanglement measures: State ordering *vs.* local operations

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A set of all states of a bi-partite quantum system can be divided into subsets each of which contains states with the same degree of entanglement. In this paper we address a question whether local operations (without classical communication) affect the entanglement-induced state ordering. We show that arbitrary unilocal channel (i.e., a channel that acts on one sub-system of a bi-partite system only) might change the ordering for an arbitrary nontrivial measure of entanglement. A slightly weaker result holds for the maximally entangled states. In particular, the maximally entangled states might not remain the most entangled ones at the output of a unilocal noise channel.

I. QUANTUM ENTANGLEMENT

Quantum phenomena (such as quantum dense coding [1], quantum teleportation [2], quantum secret sharing [3], etc.) associated with the existence of quantum entanglement represent one of the most important pillars of quantum information theory [4]. In spite of all the progress in understanding the nature of this phenomenon some features of the concept of quantum entanglement are still to be properly illuminated. In particular, due to the seminal work of Reinhard Werner [5] and others (see e.g. the review article [6]) we have a precise mathematical definition of what does it mean when we say that a bi-partite state is entangled. On the other hand a clear generally applicable operational meaning of the entanglement is still missing.

In this paper we will analyze some dynamical aspects of quantum entanglement. Specifically we will study the relation between unilocal operations and static (kinematic) properties of quantum entanglement expressed in terms of the entanglement-induced state ordering.

The concept of quantum entanglement is relatively easy to understand when we deal with pure states of bi-partite systems. This easiness originates in a close (mathematical) relationship between the concept of entanglement and the concept of statistical correlations. In fact, for pure quantum states these two concepts can be quantified by the same functions and the meaning of the statement “not entangled” is equivalent to the “not correlated”. However, conceptual differences between entanglement and statistical correlations become striking when we consider mixed states.

An important feature of quantum entanglement reflecting its behavior under local operations and classical communication has been known for some time [4]. Namely, it is well established that two (classically) communicating distant parties cannot entangle their quantum systems without performing a global operation (corresponding to some effective interaction). In other words, arbitrary local operations cannot create the entanglement even if these actions are coordinated by an exchange of classical information. Moreover, local unitary transformations do not affect the quantum entanglement at all.

These properties form a basis of our intuitive picture of quantum entanglement. Let us summarize these “natural” properties of entanglement:

- The quantum entanglement is a property of a quantum state.
- A quantum state is entangled, if it cannot be prepared from a factorized state ($\varrho_A \otimes \varrho_B$) by an action of local operations and classical communication, i.e. it cannot be expressed as a convex sum of factorized states ($\varrho_{AB} \neq \sum_j p_j \varrho_A^{(j)} \otimes \varrho_B^{(j)}$).
- LOCC (local operations plus classical communication) operations applied to an arbitrary (even entangled) quantum state can only destroy the entanglement.
- Locally unitary equivalent quantum states are equally entangled.

As we have already said, the concept of “not being entangled” is well defined. Non-entangled states are called *separable*. There is also a common agreement on the notion of maximally entangled quantum states that represent the other extreme. We say that a bi-partite quantum state is maximally entangled if it is pure and the two subsystems are in maximally mixed states, i.e. $\varrho_{AB} = |\Psi\rangle\langle\Psi|$ and $\text{Tr}_B[|\Psi\rangle\langle\Psi|] = \text{Tr}_A[|\Psi\rangle\langle\Psi|] = \frac{1}{d}I$ with $d = \min\{\dim\mathcal{H}_A, \dim\mathcal{H}_B\}$.

There are two basic questions: i) whether a given state is entangled, or not?, and ii) whether we can compare the entanglement of different quantum states. Both questions can be addressed via the so-called *entanglement measures*.

In this paper we will focus our attention on the concept of entanglement measures. We will study dynamics of entanglement under the action of local channels. Our paper is organized as follows: We start with a brief introduction to entanglement measures. Then we will analyze the stability of entanglement-induced state ordering and the properties of maximally entangled states with respect to local operations, in particular for the so-called unilocal channels. Finally, we will discuss some conceptual consequences of our analysis.

II. ENTANGLEMENT MEASURES

The entanglement (see a recent review [7]) has been identified as the key ingredient in applications such as the quantum teleportation, the quantum secret sharing, etc. However, it is also known that the presence of entanglement itself does not guarantee the success of a protocol. For instance, an arbitrary entangled state cannot be used for the teleportation. Even if a state can be exploited for this protocol the success/rate of the teleportation depends on the particular state. Hence, it seems there are states with different “quality” and “quantity” of entanglement. In order to quantify a degree of entanglement entanglement measures have been introduced. These measures are functionals defined on a state space designed to quantify the amount of entanglement in a given state. During the last ten years the topic of entanglement measures has attracted a lot of attention and many important results has been discovered.

Principally there are two approaches to the entanglement measures: i) the *operational* approach, and ii) the *axiomatic* approach. The aim of the first approach is to adopt a procedure (protocol) that crucially depends on the presence of entanglement (for example the quantum teleportation), and to quantify its success of performance depending on the particular state. Such measure would give a direct operational meaning to quantum entanglement associated with a given state. Unfortunately no such (universal) procedure is known. In the abstract axiomatic approach we reformulate our intuitive understanding of entanglement into several axioms. There exist several different (not completely equivalent) choices for the system of axioms [8], however our aim is not to discuss all these choices. We say that the functional $E : \mathcal{S}(\mathcal{H}) \rightarrow [0, \infty]$ is an entanglement measure if the following properties hold:

1. *Sharpness*: $E(\varrho_{AB}) = 0$ if and only if ϱ_{AB} is a separable state.
2. *Local unitary invariance*: $E(U_A \otimes U_B \varrho_{AB} U_A^\dagger \otimes U_B^\dagger) = E(\varrho_{AB})$ for all unitary transformations U_A, U_B and all states ϱ_{AB} .
3. *Normalization*: $E(\varrho_{AB})$ is maximal only for maximally entangled states, i.e. $E(\varrho_{AB}) = \max_{\varrho_{AB}} E(\varrho_{AB})$ if and only if $\text{Tr}_A \varrho_{AB} = \text{Tr}_B \varrho_{AB} \sim I$ and $\text{Tr} \varrho_{AB}^2 = 1$.
4. *Nonincreasing under LOCC*: A general LOCC operation transforms the original state ϱ_{AB} into a mixture of states $\omega_k^{AB} = \mathcal{E}_k^A \otimes \mathcal{E}_k^B [\varrho_{AB}]$ with probabilities p_k . This condition guarantees that the entanglement is (on average) not created by LOCC operations, i.e. $E(\varrho_{AB}) \geq \sum_k p_k E(\omega_k^{AB})$.
5. *Convexity*: $E(\sum_k p_k \omega_k^{AB}) \leq \sum_k p_k E(\omega_k^{AB})$.
6. *Additivity on pure states*: $E(\Psi_{AB} \otimes \Phi_{A'B'}) = E(\Psi_{AB}) + E(\Phi_{A'B'})$ for all pure states $\Psi_{AB}, \Phi_{A'B'}$.

The first four properties from the above list are in an agreement with our intuitive picture discussed in the previous section. In order to motivate the remaining two properties we have to take into account a situation in which a pair of systems is a part of a larger composite object. Without the loss of generality we can assume to have three parties (systems) A, B, C in a pure state Ω_{ABC} . By performing measurement on the system C and reading an outcome j (associated with the state transformation $\mathcal{I}_{AB} \otimes \mathcal{F}_j^C$) the original state $\varrho_{AB} = \text{Tr}_C \Omega_{ABC}$ is transformed into the state $\omega_j^{AB} = \text{Tr}_C \Omega_j^{ABC} = \text{Tr}_C (\mathcal{I}_{AB} \otimes \mathcal{F}_j^C) [\Omega_{ABC}]$. This happens with some probability p_j . Without the knowledge of the observed outcome j , the experimentalists possessing the systems A and B can use only the entanglement contained in the state ϱ_{AB} , because the measurement performed on C does not affect the *average* state ϱ_{AB} . However, if they acquire the information about the outcome j , they can exploit the entanglement shared in particular states ω_j^{AB} , hence they can on average exploit $\sum_j p_j E(\omega_j^{AB})$ of the entanglement. The knowledge of j cannot decrease the entanglement contained originally in ϱ_{AB} . Hence, although the measurement on the system C is a local action, the entanglement between A and B can increase, i.e. the third party can assist to A and B to increase the entanglement they share providing that the information on j is communicated to A and B . In fact, the measurements on the system C induces convex decompositions of the state $\varrho_{AB} = \sum_j p_j \omega_j^{AB}$, thus we get the convexity condition for entanglement measures.

For example, let us consider three parties A, B and C share a GHZ state $|\Omega_{ABC}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. A bipartite density operator ϱ_{AB} describes a classically maximally correlated state, which is not entangled at all and cannot be used for the teleportation. On the other hand, if the third party C performs a measurement in the dual basis $|\pm\rangle_C = \frac{1}{\sqrt{2}}|0\rangle \pm |1\rangle$ then for both outcomes ± 1 the parties A and B share a maximally entangled quantum state. In particular, $\omega_\pm^{AB} = \frac{1}{2}(|00\rangle \pm |11\rangle)(\langle 00| \pm \langle 11|)$. We see explicitly that such an assistance by the third party can significantly increase the entanglement - this is the reason for the convexity condition. Taking the maximum of average entanglement over all decompositions we obtain the so-called *entanglement of assistance* [9], $E_{\text{assist}}(\varrho_{AB}) = \max \sum_j p_j E(\omega_j^{AB})$.

The requirement of the additivity is a rather natural property of the quantum entanglement, however we lack some clear operational reason for it and it is not trivially satisfied for the measures we use. For example, the additivity of entanglement of formation is one of the most important open problems in the quantum information theory. Therefore, it is demanded that this property holds only for tensor product of pure states. In a sense this should guarantee some scaling properties of quantum entanglement, i.e. more-dimensional systems can be more entangled.

III. ORDERING VS. LOCAL OPERATIONS

Entanglement measures enable us not only to decide whether a given state is entangled, but they also allow us to conclude whether one state is more entangled than another. In fact, any entanglement measure can be used to induce an ordering on a set of quantum states. However, it has been pointed out in Ref. [10] and analyzed by many others [11] that entanglement-induced orderings for two different entanglement measures E_1, E_2 can differ. Even for the most commonly used measures of entanglement [11] there exists a pair of states ω_{AB} and ϱ_{AB} such that $E_1(\varrho_{AB}) > E_1(\omega_{AB})$, but $E_2(\varrho_{AB}) < E_2(\omega_{AB})$.

In Ref. [12] we addressed the question whether for a given entanglement measure the ordering is preserved under the action of local operations (without a classical communication). In a sense, we postulated an additional axiom that should be fulfilled by a “good” entanglement measure. There are several proposals for entanglement measures satisfying the basic properties 1-4 from the above list. For example, the entanglement of formation [13], the concurrence [14], tangle [15], the relative entropy of entanglement [16], the negativity [17], the squashed entanglement [18], etc. Certainly, the practical computability might be a non-trivial problem. In most cases the optimization and the minimization can be accomplished only numerically. For a two-qubit system the entanglement of formation $E_f = \inf \sum_j p_j \tilde{S}_{vN}(\Psi_j)$, the tangle $\tau = \inf \sum_j p_j \tilde{S}_L(\Psi_j)$ and the concurrence $C = \sqrt{\tau}$ are mutually closely related and they are straightforward to compute. We used the notation \tilde{S} for the corresponding entropy S of the reduced state $\omega = \text{Tr}_B \Psi$. The infima are taken over all convex decompositions of the given state ϱ into pure states $\{\Psi_j\}$. The indexes vN, L stand for the von Neumann entropy ($S_{vN} = -\text{Tr} \varrho \log \varrho$) and the linear entropy $S_L = 2(1 - \text{Tr} \varrho^2)$, respectively. It was shown in [14] that for two qubits $E_f = h(\frac{1}{2}[1 + \sqrt{1 - \tau}])$, $\tau = C^2$ and $C = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}$, where λ_j are decreasingly ordered eigenvalues of the matrix $R = \varrho(\sigma_y \otimes \sigma_y) \varrho^*(\sigma_y \otimes \sigma_y)$ and $h(x) = -x \log x - (1 - x) \log(1 - x)$ is the binary entropy.

In our previous work [12] we have shown that a stability of the entanglement-induced ordering is not compatible with the listed axioms. A simple counter-example one can present involves four qubits divided into two groups. Moreover, we have explicitly shown that the ordering is not preserved for all two-qubit measures providing one of the subsystems is affected by the depolarizing channel $\mathcal{E}_p[\omega] = p\omega + (1 - p)\frac{1}{2}I$. The violation of the ordering is depicted in the diagram on Fig. 1. Based on this explicit counter-example we can argue that there is no (nontrivial) entanglement measure E that is stable under the action of local operations of the form $\mathcal{E} \otimes \mathcal{I}$, where \mathcal{E} is a tracepreserving completely positive linear map on the system A only.

Let us consider the so-called unilocal channel of the form $\mathcal{E} \otimes \mathcal{I}$ and some entanglement measure E . The

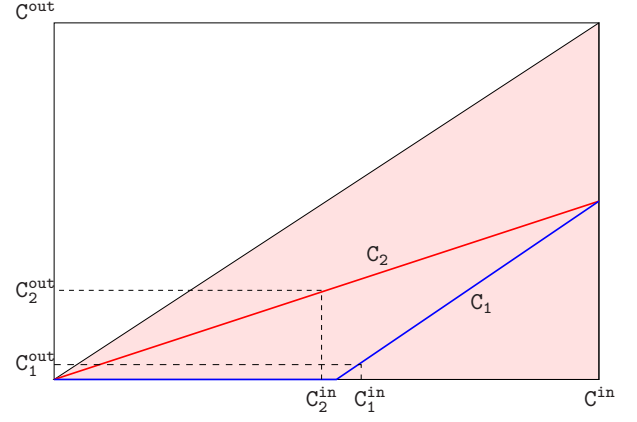


FIG. 1: The input/output diagram for the concurrence for two families of states: 1) Werner states $\varrho_2 = q\Psi_+ + (1 - q)\frac{1}{4}I$ with Ψ_+ being a projector onto maximally entangled state $|\psi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, and 2) pure states $\varrho_1 = |\psi\rangle\langle\psi|$ with $|\psi\rangle = \alpha|00\rangle + \beta|11\rangle$. We consider the depolarizing channel with $p = 1/2$. The states from the counterexample discussed in the paper are displayed and the change in the ordering is visible. The region under the line $C^{\text{out}} = C^{\text{in}}$ represents the allowed region that is achievable by local channels. The concurrence is measured in dimensionless units.

action of such local channel can be expressed in the $[E_{\text{in}}, E_{\text{out}}]$ -diagram with respect to a given measure of the entanglement E . Whenever we find that for fixed values of E_{out} there exist more input values E_{in} , one can easily construct a suitable counter-example violating the condition of the ordering-preservation

$$E(\varrho_1) > E(\varrho_2) \Rightarrow E(\varrho'_1) \geq E(\varrho'_2) \quad (3.1)$$

valid for all states ϱ_1, ϱ_2 and $\varrho'_j = \mathcal{E} \otimes \mathcal{I}[\varrho_j]$ ($j = 1, 2$).

More specifically. Let us define a “horizontal fiber” $\mathcal{F}_h(E_{\text{out}})$ to be a set of all values of E_{in} such that there exists a state ϱ_{in} with $E(\varrho_{\text{in}}) = E_{\text{in}}$ and $E(\mathcal{E} \otimes \mathcal{I}[\varrho_{\text{in}}]) = E_{\text{out}}$. Whenever $\mathcal{F}_h(E_{\text{out}}) \cap \mathcal{F}_h(E'_{\text{out}}) \neq \emptyset$ for all pairs of possible values $E_{\text{out}}, E'_{\text{out}}$ and $\mathcal{F}_h(E_{\text{out}}) \neq \mathcal{F}_h(E'_{\text{out}})$, the counter-example can be designed. Consider $E_{\text{out}} > E'_{\text{out}}$. Because of the nonempty intersection of $\mathcal{F}_h(E_{\text{out}}), \mathcal{F}_h(E'_{\text{out}})$, there are states ϱ_{in}^j ($j = 1, 2$) with the same amount of the initial entanglement $E_{\text{in}}^1 = E_{\text{in}}^2$, but different values of the final entanglement $E_{\text{out}} = E_{\text{out}}^1 > E_{\text{out}}^2 = E'_{\text{out}}$. Moreover, it is possible to choose ϱ_{in}^1 and ϱ_{in}^2 to have different values of entanglement so that the ordering is not preserved, in particular, $E(\varrho_{\text{in}}^1) < E(\varrho_{\text{in}}^2)$. Each unilocal channel \mathcal{E} determines a set $S_{\mathcal{E}}$ in the $[E_{\text{in}}, E_{\text{out}}]$ -diagram. In particular, for $S_{\mathcal{E}}$ forming some region (i.e. two-dimensional geometrical object) the ordering is not preserved, because there are values $E_{\text{out}}, E'_{\text{out}}$ for which $\mathcal{F}_h(E_{\text{out}}) \cap \mathcal{F}_h(E'_{\text{out}}) \neq \emptyset$.

The formal description presented in the above paragraph as well as the particular analysis itself might be technically difficult. In fact, the illustration of the set $S_{\mathcal{E}}$ requires to evaluate the entanglement for all possible states. However, intuitively the situation expressed in

Fig. 1 is not that complicated.

The observation that deserves special attention is that in order to avoid the counter-examples of the above form the entanglement measure and the transformation \mathcal{E} must have very specific (and very peculiar) properties that are reflected in the $[E_{\text{in}}, E_{\text{out}}]$ -diagram. If the possible values of E_{out} form a continuum (which is the case for all the measures we use), then the corresponding set $S_{\mathcal{E}}$ must form a line. But this means, that either the equally entangled states are always mapped into the equally entangled states, or $S_{\mathcal{E}}$ consists of horizontal and vertical lines. The corresponding maps would be indeed interesting.

We started our discussion with the question whether there exists an entanglement measure such that for *all* channels $\mathcal{E} \otimes \mathcal{I}$ the induced ordering is preserved. However, the analysis led us to another questions. Specifically, for which channels a given entanglement measure is preserved? Our conjecture is that essentially arbitrary local channel affects the ordering. The only known exceptions are: 1) a unitary channels ($E_{\text{out}} = E_{\text{in}}$), 2) and the entanglement-breaking channels ($E_{\text{out}} = 0$). Other “entanglement-order-preserving” channels would be of interest *per se*. There is a strong evidence that such channels do not exist. Consequently, it seems that the measures stable under local operations should be discrete, i.e. the entanglement can achieve only certain countable set of values. An example of such measure is the trivial δ -measure that answers the question whether a given state is entangled, or not. Our statement holds modulo this type of “discrete” entanglement measures.

IV. MAXIMAL ENTANGLEMENT VS. LOCAL OPERATIONS

It is important to know how the entanglement behaves under the action of quantum dynamics [19]. For example, it is interesting to know whether local sources of decoherence are relevant for a given quantum protocol based on entangled states. In the previous section we have analyzed how the local operations affect the entanglement-induced ordering. Positive answer to such question would give us a strong tool how to analyze the effect of local noise in general settings just by analyzing the behavior of the most entangled states. Unfortunately, we have found that the situation is puzzling, because it seems that essentially arbitrary unilocal channel does not preserve the ordering whatever measure we choose. In this section we will focus on a simpler question: How much can we learn from the analysis of the dynamics of maximally entangled states?

In Ref. [12] we concluded that maximally entangled state remains most entangled also after the application of the local transformation $\mathcal{E} \otimes \mathcal{I}$. Unfortunately, this statement is not correct and there is a loophole in the proof [21, 22]. Here is a simple counter-example. Consider a system consisting of four qubits (the qubits A, A' on one side and B, B' on the other one) and a local

map $\mathcal{E}_{AA'} = \mathcal{P}_0 \otimes I + \mathcal{P}_1 \otimes \mathcal{A}$, where \mathcal{P}_j is defined as $\mathcal{P}_j[X] = P_j X P_j$ ($P_j = |j\rangle\langle j|$), and $\mathcal{A}[X] = \frac{1}{2}\text{Tr}(X)I$. This transformation “checks” the state of A and either leaves A' unaffected, or it contracts its state into a maximally mixed state. We will analyze the action of such channel on two states: 1) $\varrho_1 = \rho_{ABA'B'} = |0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B \otimes P_{A'B'}^+$, or 2) maximally entangled state $\varrho_2 = P^+ = P_{ABA'B'}^+ = P_{AB}^+ \otimes P_{A'B'}^+$, where P_{AB}^+ is a projector onto a maximally entangled state of qubits A, B , and similarly for $P_{A'B'}^+$. The first of these states is invariant under the action of $\mathcal{E}_{AA'} \otimes \mathcal{I}_{BB'}$, i.e. $\varrho'_1 = \varrho_1$, but $\varrho'_2 = \mathcal{E}_{AA'} \otimes \mathcal{I}_{BB'}[P^+] = \frac{1}{2}\varrho_1 + \frac{1}{2}|1\rangle\langle 1|_A \otimes |1\rangle\langle 1|_B \otimes \frac{1}{4}I$. The state ϱ'_2 is, if entangled, for sure is strictly less entangled than ϱ'_1 , i.e. ordering is not preserved for an arbitrary measure of entanglement. The convexity guarantees that $E(\varrho'_2) \leq \frac{1}{2}E(\varrho'_1) < E(\varrho'_1)$.

This result suggests that it is not straightforward to see how much the analysis of dynamics of maximally entangled states can tell us about the entanglement dynamics in general. On the other hand, in spite of the result related to the entanglement-induced ordering, in the present case the maximality is preserved for larger class of channels. Their characterization is an open problem and will be analyzed elsewhere [22]. An interesting feature that remains valid is that all maximally entangled states are (under unilocal channels) mapped into states with the same amount of entanglement [12]. This holds for any measure of entanglement.

V. SPECULATIONS AND CONCLUSIONS

As a result of our analysis we discovered new features and properties of entanglement measures. We found that the ordering that implies statements such as “one state is more/less entangled than another” is not preserved under the action of local operations. Moreover, such ordering is affected by all unilocal operations except the unitary and the entanglement-breaking channels. Surprisingly enough, we also found that the maximally entangled states might be more fragile than “less” entangled states. This might sound counter-intuitive, but in some realistic cases, in which the systems are affected by a local noise, it could be better to start with less (noise-dependent) entangled state in order to increase the success of the protocol. Hence, the operational meaning of the property “being more/less entangled” is questionable. Operationally, “more entangled” should be synonymous to “having larger rate” of success. However, just a small modification of protocols (e.g. taking into account a local noise) might change this interpretation. Hence, does it make any sense to use the entanglement measures for the state ordering? If not, then what are these measures good for?

Entanglement measures still provide us with very powerful tools enabling us to decide the basic question, whether a given state is entangled, or not. In fact, it is much simpler to compute the concurrence of two qubits

than to prove the (non)existence of a separable decomposition. It might be that the idea of entanglement-induced state ordering cannot be based on some entanglement measure. To introduce such concept one should probably adopt different approach, in which the stability with respect to local operations is fulfilled “by the definition”. Even in this case we have more options depending on the class of operations we will consider. We can say that a state ω_1 is more, or equally entangled than a state ω_2 ($\omega_1 \succeq \omega_2$) if and only if there exists a completely positive tracepreserving linear operation $\mathcal{E}_A \otimes \mathcal{E}_B$ such that $\omega_2 = \mathcal{E}_A \otimes \mathcal{E}_B[\omega_1]$. This is compatible with the fact, that entanglement can be only decreased by the action of local operations (LO). Alternatively, one can use the class of LOCC operations, or stochastic LOCC (SLOCC) operations. Two states are equally entangled if $\omega_1 \succeq \omega_2$ and $\omega_2 \succeq \omega_1$ simultaneously. If two states are not equivalent, but $\omega_2 \succeq \omega_1$, then $\omega_2 \succ \omega_1$. All these types of entanglement-based orderings are, in principle, partial, i.e. not all states are comparable. For example, using the SLOCC-ordering all two-qubit entangled states are equally entangled, because they can be used for the teleportation. The LOCC-ordering is more strict and for the LO-ordering pure states with different Schmidt coefficients are not comparable. Intuitively, the most physical/operational is the LOCC-based state ordering.

Recently, Kinoshita et al. in [23] analyzed compatibility of the LOCC-based ordering under the action of local operations. They presented an example of two states $\omega_1 \succ \omega_2$ that are transformed by a unilocal operation $\mathcal{E} \otimes \mathcal{I}$ (the so-called selective entanglement-breaking channels) into ω'_1, ω'_2 such that $\omega'_2 \succ \omega'_1$. This explicit example supports our conclusion about the existence of entanglement-induced state orderings compatible with local operations, because it shows that for an arbitrary entanglement measure satisfying the LOCC monotonicity condition the entanglement-induced ordering is not preserved. But, one can make even stronger conclusion that also the “operational” LOCC-based state ordering is not robust with respect to local operations. It seems that there is no way how to introduce a nontrivial entanglement-related state ordering compatible with local operations. The only option is to use the trivial δ -measure, or some simple modification of it.

In the analysis of entanglement dynamics it is of interest to specify times at which the entanglement disappears. Although any particular dynamics depends on the initial state, these “entanglement-breaking” time instants t_{sep} can be completely characterized by the analysis of the maximally entangled state. The channel is called entanglement-breaking \mathcal{E} if and only if $\omega' = \mathcal{E} \otimes \mathcal{I}[\omega]$ is separable for all initial states ω . It is sufficient to verify this property for a maximally entangled state, i.e. whether $E(\mathcal{E} \otimes \mathcal{I}[P_+]) = 0$ [24]. The local dynamics is given by a one-parametric set of completely positive maps \mathcal{E}_t . We have analyzed [20] the general qubit

master equation generating semigroup dynamics. The qubit semigroup evolution is characterized by two time scales: the decoherence time $T_{\text{decoherence}}$ and the decay time T_{decay} . What are the limits on the entanglement decay? Which process is the fastest one? These questions are not answered in [20], but all the necessary tools are derived in that paper. It is known that in some cases $t_{sep} \rightarrow \infty$, but what is the shortest possible decay time t_{sep} ? The result is that there is no limit and t_{sep} can be arbitrarily small. For example, under the action of a local depolarization process $\mathcal{E}_t[\rho] = e^{-t/T}\rho + (1 - e^{-t/T})\frac{1}{2}I$ the maximally entangled states evolves into the state $\omega_t = e^{-t/T}P_+ + (1 - e^{-t/T})\frac{1}{4}I$ (Werner states). Hence, the entanglement vanishes for $t_{sep} = T \ln 3$. The parameter T can be adjusted so that the entanglement is destroyed in arbitrarily small time t_{sep} . In general, the vanishing decoherence rate guarantees the shortest possible entanglement decay time, i.e. the process of entanglement decay can be “infinitely” fast.

Let us get back to the status of entanglement measures. The main message of this contribution is that the quantification of entanglement based on entanglement measures define a state ordering that is not preserved under the action of local operations. The interpretation of these measures should be reconsidered. It seems that large values of entanglement measures characterize the “distance” from the set of maximally entangled states, which is clearly defined. Similarly, small values should correspond to states that are very far (in the sense of entanglement) from the maximally entangled ones and very close to the separable region of the state space. The particular mathematical forms of these statements is not known, but the meaning of entanglement degree could be hidden there. The axiomatic entanglement measures can quantify different aspects of quantum entanglement, or they can serve as bounds for particular protocols. To understand the entanglement itself it is important to understand the numbers we use to quantify this phenomenon. Thinking about the relation between the state ordering, the entanglement measures, and the robustness with respect to local operations, opens new interesting conceptual questions deserving a deeper investigation.

Acknowledgements

We would like to thank to Marco Piani for pointing out that there is a wrong statement in our earlier paper [12]. We thank him for interesting and encouraging discussions. This work was supported in part by the European Union projects QAP, CONQUEST, INTAS project number 04-77-7289, and by the Slovak Academy of Sciences via the project CE-PI, and by the projects APVT-99-012304 and GAČR GA201/01/0413.

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- [1] C.H. Bennett and S.J. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).
 - [2] C.H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Perez, and W.K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
 - [3] M. Hillery, V. Bužek, and A. Berthiaume, Phys. Rev. A **59**, 1829 (1999).
 - [4] M.A. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information*, (Cambridge University Press, Cambridge, 2000).
 - [5] R.F. Werner, Phys. Rev. A **40**, 4277 (1989).
 - [6] M.B. Plenio and S.S. Virmani, in *Lectures on Quantum Information*, eds. D.Bruss and G.Leuchs (Wiley, Weinheim, 2007), p. 161 (2007).
 - [7] M.B. Plenio and S.S. Virmani, Quant. Inf. Comp. **7**, 1 (2007); see also quant-ph/0504163.
 - [8] M.J. Donald, M. Horodecki, and O. Rudolph, J. Math. Phys. **43**, 4252 (2002).
 - [9] D.P. DiVincenzo, *The entanglement of assistance* in Lecture Notes in Computer Science 1509 (Springer-Verlag, Berlin, 1999), pp. 247-257.
 - [10] J. Eisert and M.B. Plenio, J. Mod. Opt. **46**, 145 (1999).
 - [11] A. Miranowicz and A. Grudka, Phys. Rev. A **70**, 032326 (2004).
 - [12] M. Ziman and V. Bužek, Phys. Rev. A **73**, 012312 (2006).
 - [13] C.H. Bennett, D.P. DiVincenzo, J. Smolin, and W.K. Wootters, Phys. Rev. A **54**, 3824 (1996).
 - [14] W.K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
 - [15] V. Coffmann, J. Kundu, and W.K. Wootters, Phys. Rev. A **61**, 052306 (2000).
 - [16] V. Vedral, M.B. Plenio, M.A. Rippin, and P.L. Knight, Phys. Rev. Lett. **78**, 2275 (1997).
 - [17] G. Vidal and R.F. Werner, Phys. Rev. A **65**, 032314 (2002), see also quant-ph/0102117.
 - [18] M. Christandl and A. Winter, J. Math. Phys. **45**, 829 (2004).
 - [19] K. Zyczowski, P. Horodecki, M. Horodecki, and R. Horodecki, Phys. Rev. A **65**, 012101 (2002).
 - [20] M. Ziman and V. Bužek, Phys. Rev. A **72**, 052325 (2005).
 - [21] M. Piani, *private communication*
 - [22] M. Piani and M. Ziman, *in preparation*
 - [23] Y. Kinoshita, R. Namiki, T. Yamamoto, M. Koashi, and N. Imoto, *Selective entanglement breaking*, quant-ph/0611171
 - [24] M. Horodecki, P.W. Shor, and M.B. Ruskai, *General Entanglement Breaking Channels*, Rev. Math. Phys **15**, 629–641 (2003), quant-ph/0302031